HARMONIC DISTRIBUTIONS AND CONFORMAL DEFORMATIONS

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Abstract. We may consider a distribution on a Riemannian manifold as a section of a Grassmann bundle. Therefore we may speak about harmonicity of a distribution. We compare harmonicity of a distribution with respect to conformal deformation of a Riemannian metric. As a special case we consider distributions on manifolds of constant curvature.

1. Introduction

Let (M,g) be a Riemannian manifold, σ a p–dimensional distribution on M. Then we may consider σ as a section of a Grassmann bundle $G_p(M)$ of p–dimensional subspaces of tangent spaces of M. Since $G_p(M)$ is a homogeneous bundle, Riemannian metric g induces a Riemannian metric g^S on $G_p(M)$. Thus we may speak about harmonicity of a distribution σ as of harmonicity of a map $\sigma:(M,g)\to(G_p(M),g^S)$ between Riemannian manifolds, see [2] and [3]. When we consider oriented distributions, we may identify σ at a point $x\in M$ with a p–vector $e_1\wedge\ldots\wedge e_p$, where $e_1,\ldots,e_n$ is an oriented orthonormal frame at x adapted to σ, see [5] for details. This leads to a map $\sigma:M\to\Lambda^p(M)$. Since $\Lambda^p(M)$, as a fibre bundle with bundle metric, carries a Riemannian metric induced from g, again we may consider harmonicity of a distribution. These two ways lead to the same condition of harmonicity.

Harmonic distributions, in above meaning, have been recently considered by many authors. In [3] authors deal with distributions on Lie groups, in [5] with Hopf distribution, whereas in [6] harmonicity of distributions on locally conformal Kähler manifolds was studied. One dimensional foliations were considered in [4].

In this paper we obtain some results about harmonicity of distributions with respect to conformal deformation of a Riemannian metric. The method of changing the metric on domain or on codomain was considered for example in [7] to obtain biharmonic non–harmonic mappings. We show that

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in some cases vertical harmonicity is a conformal invariant. Moreover, we consider distributions on manifolds of constant curvature.

2. Tension field of a distribution

Let (M,g) be a n–dimensional Riemannian manifold. Consider the orthonormal frame bundle $\xi : O(M) \to M$. Take $p < n$ and put $G = O(n)$, $H = O(p) \times O(n-p)$. Then the Grassmann bundle $G_p(M)$ is the associated bundle $O(M) \times_G (G/H)$. Let π denote the projection in this bundle. Moreover let $\zeta : O(M) \to G_p(M)$ be defined in the following way, $\zeta(e_1,\dots,e_n) = \mathrm{Span}(e_1,\dots,e_p)$. Then $\pi \circ \zeta = \xi$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Put

$$\mathfrak{m} = \{ \begin{bmatrix} 0 & A \\ -A^\top & 0 \end{bmatrix} \mid \quad A \text{ is any } (n-p) \times p \text{ martix} \}.$$

Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Let $\langle \cdot, \cdot \rangle$ denote a G–invariant metric on G/H or equivalently $\mathrm{Ad}_G(H)$–invariant inner product on $\mathfrak{m}$. Let ω be a connection form of a connection in $O(M)$. Then

$$TG_p(M) = \mathcal{V} \oplus \mathcal{H},$$

where $\mathcal{V} = \ker \pi_*$ and $\mathcal{H} = \zeta_*(\ker \omega)$. Since $\mathfrak{m}$ is isomorphic to the fibre $\mathcal{V}_x$ (denote this isomorphism by φ), we may define a Riemannian metric g^S on $G_p(M)$ as

$$g^S(V,W) = g(\pi_* V, \pi_* W) + \langle \varphi^{-1} V, \varphi^{-1} W \rangle, \quad V, W \in TG_p(M).$$

For details see [8] and [3].

Throughout the paper we use the index convention

$$1 \le \alpha, \beta, \gamma \le n, \quad 1 \le a, b, c \le p, \quad p+1 \le i, j, k \le n.$$

Let σ be a p–dimensional distribution on M. Then we may consider σ as a section of the Grassmann bundle $G_p(M)$. Thus harmonicity of σ is well defined. We say that σ is *harmonic* (with respect to g) if its tension field $\tau(\sigma)$ vanishes

$$\tau(\sigma) = \mathrm{tr} \nabla \sigma_* = 0,$$

where ∇ is a connection induced from Levi–Civita connection on M and pull–back connection on the pull–back bundle $\sigma^{-1} TG_p(M)$. For more details on harmonic maps see [1]. Let $\sigma^\perp$ denotes the distribution orthogonal to σ, so that we have an orthogonal decomposition $TM = \sigma \oplus \sigma^\perp$. Hence every vector $X \in T_x M$ has a unique decomposition $X = X^\top + X^\perp$, $X^\top \in \sigma(x)$,

$X^\perp \in \sigma^\perp(x)$. By [3, Proposition 2 and 3] σ is harmonic if and only if

$$\sum_{\alpha,a} R((\nabla_{e_\alpha} e_a)^\perp, e_a)e_\alpha = 0, \tag{2.1}$$

$$\sum_{\alpha} (\nabla_{e_\alpha}(\nabla_{e_\alpha} e_a)^\perp - \nabla_{e_\alpha}(\nabla_{e_\alpha} e_a)^\top - \nabla_{\nabla_{e_\alpha} e_\alpha} e_a)^\perp = 0, \quad \text{for every } a, \tag{2.2}$$

where R is a curvature tensor, (e_α) is an orthonormal frame such that $e_a \in \Gamma(\sigma)$. Moreover, we say that σ is *vertically harmonic* (resp. *horizontally harmonic*) if (2.1) (resp. (2.2)) holds. Notice that conditions (2.1) and (2.2) can be written in the form

$$\sum_{a,b} R((\nabla_{e_b} e_a)^\perp, e_a)e_b + \sum_{i,j} R((\nabla_{e_i} e_j)^\top, e_j)e_i = 0 \tag{2.1'}$$

and for every a and i

$$\begin{aligned} 0 &= \sum_b g((\nabla^2 e_a)(e_b, e_b), e_i) - \sum_j g((\nabla^2 e_i)(e_j, e_j), e_a) \\ &+ 2\sum_{b,c} g(e_a, \nabla_{e_b} e_c) g(\nabla_{e_b} e_c, e_i) - 2\sum_{j,k} g(e_a, \nabla_{e_j} e_k) g(\nabla_{e_j} e_k, e_i). \end{aligned} \tag{2.2'}$$

where $(\nabla^2 X)(Y,Z) = \nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X$. Indeed,

$$\begin{aligned} 0 &= \sum_{\alpha,a} R((\nabla_{e_\alpha} e_a)^\perp, e_a)e_\alpha \\ &= \sum_{a,b} R((\nabla_{e_b} e_a)^\perp, e_a)e_b + \sum_{a,i} R((\nabla_{e_i} e_a)^\perp, e_a)e_i \\ &= \sum_{a,b} R((\nabla_{e_b} e_a)^\perp, e_a)e_b + \sum_{a,i,j} g(\nabla_{e_i} e_a, e_j) R(e_j, e_a)e_i \\ &= \sum_{a,b} R((\nabla_{e_b} e_a)^\perp, e_a)e_b + \sum_{a,i,j} g(\nabla_{e_i} e_j, e_a) R(e_a, e_j)e_i \\ &= \sum_{a,b} R((\nabla_{e_b} e_a)^\perp, e_a)e_b + \sum_{i,j} R((\nabla_{e_i} e_j)^\top, e_j)e_i. \end{aligned}$$

As for the second equality, we have for any a

$$\begin{aligned} 0 &= \sum_\alpha (\nabla_{e_\alpha}(\nabla_{e_\alpha} e_a)^\perp - \nabla_{e_\alpha}(\nabla_{e_\alpha} e_a)^\top - \nabla_{\nabla_{e_\alpha} e_\alpha} e_a)^\perp \\ &= \sum_\alpha (\nabla_{e_\alpha} \nabla_{e_\alpha} e_a - 2\nabla_{e_\alpha}(\nabla_{e_\alpha} e_a)^\top - \nabla_{\nabla_{e_\alpha} e_\alpha} e_a)^\perp \\ &= \sum_\alpha ((\nabla^2 e_a)(e_\alpha, e_\alpha) - 2\nabla_{e_\alpha}(\nabla_{e_\alpha} e_a)^\top)^\perp \end{aligned}$$

Thus for any a and i

$$\begin{aligned}
0 &= \sum_\alpha g((\nabla^2 e_a)(e_\alpha, e_\alpha) - 2\nabla_{e_\alpha}(\nabla_{e_\alpha} e_a)^\top, e_i) \\
&= \sum_b g((\nabla^2 e_a)(e_b, e_b), e_i) + \sum_j g((\nabla^2 e_a)(e_j, e_j), e_i) \\
&\quad + 2\sum_b g((\nabla_{e_b} e_a)^\top, \nabla_{e_b} e_i) + 2\sum_j g((\nabla_{e_j} e_a)^\top, \nabla_{e_j} e_i).
\end{aligned}$$

Since

$$g(\nabla_{e_j}\nabla_{e_j} e_a, e_i) = -2g(\nabla_{e_j} e_a, \nabla_{e_j} e_i) - g(e_a, \nabla_{e_j}\nabla_{e_j} e_i),$$

then

$$g((\nabla^2 e_a)(e_j, e_j), e_i) = -g((\nabla^2 e_i)(e_j, e_j), e_a) - 2g(\nabla_{e_j} e_a, \nabla_{e_j} e_i).$$

Therefore, for any a and i

$$\begin{aligned}
0 &= \sum_b g((\nabla^2 e_a)(e_b, e_b), e_i) - \sum_j g((\nabla^2 e_i)(e_j, e_j), e_a) \\
&\quad - 2\sum_j g(\nabla_{e_j} e_a, \nabla_{e_j} e_i) + 2\sum_b g((\nabla_{e_b} e_a)^\top, \nabla_{e_b} e_i) \\
&\quad + 2\sum_j g(\nabla_{e_j} e_a, \nabla_{e_j} e_i) - 2\sum_j g((\nabla_{e_j} e_a)^\perp, \nabla_{e_j} e_i) \\
&= \sum_b g((\nabla^2 e_a)(e_b, e_b), e_i) - \sum_j g((\nabla^2 e_i)(e_j, e_j), e_a) \\
&\quad + 2\sum_{b,c} g(\nabla_{e_b} e_a, e_c) g(e_c, \nabla_{e_b} e_i) - 2\sum_{j,k} g(\nabla_{e_j} e_a, e_k) g(e_k, \nabla_{e_j} e_i) \\
&= \sum_b g((\nabla^2 e_a)(e_b, e_b), e_i) - \sum_j g((\nabla^2 e_i)(e_j, e_j), e_a) \\
&\quad + 2\sum_{b,c} g(e_a, \nabla_{e_b} e_c) g(\nabla_{e_b} e_c, e_i) - 2\sum_{i,k} g(e_a, \nabla_{e_j} e_k) g(\nabla_{e_j} e_k, e_i).
\end{aligned}$$

Denote the left hand side of (2.1') by $\tau^h_g(\sigma)$ and the right hand side of (2.2') by $\tau^v_g(\sigma)_{a,i}$. We call $\tau^h_g(\sigma)$ a *horizontal tension field* and $\tau^v_g(\sigma)_{a,i}$ a *vertical tension field* with respect to g. Above formulas imply

Proposition 2.1. [2, 5] *A distribution σ is harmonic (resp. vertically, resp. horizontally harmonic) if and only if $\sigma^\perp$ is harmonic (resp. vertically, resp. horizontally harmonic).*

3. Main results

Let (M, g) be a Riemannian manifold, σ a p–dimensional distribution on M. Let $\sigma^\perp$ be the distribution orthogonal to σ. We define the *unsymmetrized*

second fundamental form A^σ and *symmetrized second fundamental form* B^σ by

$$A^\sigma_X Y = (\nabla_X Y)^\perp,$$
$$B^\sigma(X,Y) = \frac{1}{2}(A^\sigma_X Y + A^\sigma_Y X), \quad X,Y \in \Gamma(\sigma).$$

In the case that σ is integrable, A^σ is symmetric and $B^\sigma = A^\sigma$. If $B^\sigma = 0$, then we say that σ is *totally geoedsic*. The *mean curvature* of σ is a vector field

$$H^\sigma = \mathrm{tr} B^\sigma = \sum_a (\nabla_{e_a} e_a)^\perp,$$

where $e_1, \ldots, e_p$ is an orthonormal frame for σ. A distribution σ is called *minimal* if $H^\sigma = 0$. Moreover, we put

$$H = H^\sigma + H^{\sigma^\perp}$$

and define the Ricci tensor with respect to σ to be a $(1,1)$–tensor of the form $\mathrm{Ric}_\sigma(X) = \sum_a R(X, e_a)e_a$.

Proposition 3.1. *Let σ be a distribution on a Riemannian manifold (M,g) of nonzero constant curvature κ. Then*

$$\tau^h_g(\sigma) = \kappa H. \tag{3.1}$$

Therefore σ (and hence $\sigma^\perp$) is horizontally harmonic if and only if σ and $\sigma^\perp$ are minimal.

Proof. The curvature tensor R is of the form

$$R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y).$$

Thus by (2.1') we get

$$\begin{aligned}\tau^h_g(\sigma) &= \kappa(\sum_{a,b} \delta^a_b (\nabla_{e_b} e_a)^\perp - \sum_{a,b} g((\nabla_{e_b} e_a)^\perp, e_b) e_a) \\ &+ \kappa(\sum_{i,j} \delta^i_j (\nabla_{e_i} e_j)^\top - \sum_{i,j} g((\nabla_{e_i} e_j)^\top, e_i) e_j) \\ &= \kappa H,\end{aligned}$$

hence $\tau^h_g(\sigma) = 0$ if and only if $H^\sigma = H^{\sigma^\perp} = 0$. □

Proposition 3.2. *Let σ and $\sigma^\perp$ be totally geodesic orthogonal foliations on a Riemannian manifold. Then both are harmonic.*

Proof. A^σ and $A^{\sigma^\perp}$ vanish. Thus

$$(\nabla_{e_a} e_b)^\perp = 0, \quad (\nabla_{e_i} e_j)^\top = 0.$$

Hence conditions (2.1') and (2.2') hold, so σ is harmonic. By Proposition 2.1 $\sigma^\perp$ is harmonic. □

Consider now a Riemannian metric $\tilde{g} = e^{2\mu} g$, where μ a smooth function on M. Let $\nabla, \tilde{\nabla}$ and $R, \tilde{R}$ denote the Levi–Civita connections and curvature tensors of g and $\tilde{g}$, respectively. We define *hessian operator* Hess_μ, *hessian* hess_μ and *laplacian* $\Delta\mu$ of a function μ as

$$\begin{aligned} \mathrm{Hess}_\mu(X) &= \nabla_X \nabla\mu, \\ \mathrm{hess}_\mu(X,Y) &= g(\mathrm{Hess}_\mu(X), Y), \quad X, Y \in \Gamma(TM), \\ \Delta\mu &= \mathrm{tr}(\mathrm{Hess}_\mu). \end{aligned}$$

For any $X, Y, Z \in \Gamma(TM)$ one may check that the Levi–Civita connection and curvature tensor of $\tilde{g}$ are

$$\tilde{\nabla}_X Y = \nabla_X Y + (Y\mu)X + (X\mu)Y - g(X,Y)\nabla\mu \tag{3.2}$$

and

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z - g(Y,Z)\mathrm{Hess}_\mu(X) + g(X,Z)\mathrm{Hess}_\mu(Y) \\ &\quad + \big((Y\mu)(Z\mu) - g(Y,Z)|\nabla\mu|^2 - \mathrm{hess}_\mu(Y,Z)\big)X \\ &\quad - \big((X\mu)(Z\mu) - g(X,Z)|\nabla\mu|^2 - \mathrm{hess}_\mu(X,Z)\big)Y \\ &\quad + \big((X\mu)g(Y,Z) - (Y\mu)g(X,Z)\big)\nabla\mu \end{aligned} \tag{3.3}$$

Moreover, we put

$$\Delta_\sigma = \mathrm{tr}_\sigma(\mathrm{Hess}_\mu) = \sum_a \mathrm{hess}_\mu(e_a, e_a),$$

where e_a is an orthonormal frame of σ.

Assume $\dim M = n$ and put $q = n - p = \dim\sigma^\perp$. We compute horizontal and vertical tension field of σ with respect to $\tilde{g}$.

Theorem 3.3. (1) *Horizontal tension fields of σ with respect to g and $\tilde{g}$ are related as follows*

$$\begin{aligned} e^{4\mu}\tau^h_{\tilde{g}}(\sigma) &= \tau^h_g(\sigma) - \mathrm{Ric}_\sigma((\nabla\mu)^\perp) - \mathrm{Ric}_{\sigma^\perp}((\nabla\mu)^\top) \\ &\quad - \mathrm{Hess}_\mu(H) - |\nabla\mu|^2 H + g(\nabla\mu, H)\nabla\mu \\ &\quad + ((p-q)|(\nabla\mu)^\top|^2 + \Delta_\sigma\mu)(\nabla\mu)^\perp \\ &\quad + ((q-p)|(\nabla\mu)^\perp|^2 + \Delta_{\sigma^\perp}\mu)(\nabla\mu)^\top \\ &\quad + p\mathrm{Hess}_\mu((\nabla\mu)^\perp) + q\mathrm{Hess}_\mu((\nabla\mu)^\top) \\ &\quad + (\nabla_{(\nabla\mu)^\top}(\nabla\mu)^\perp)^\top - (\nabla_{(\nabla\mu)^\perp}(\nabla\mu)^\top)^\top \\ &\quad + (\nabla_{(\nabla\mu)^\perp}(\nabla\mu)^\top)^\perp - (\nabla_{(\nabla\mu)^\top}(\nabla\mu)^\perp)^\perp \\ &\quad - \mathrm{tr}(\nabla_*(\nabla_*\nabla\mu)^\perp)^\top - \mathrm{tr}(\nabla_*(\nabla_*\nabla\mu)^\top)^\perp. \end{aligned} \tag{3.4}$$

(2) *Vertical tension fields of σ with respect to g and $\tilde{g}$ are related as follows*

$$
\begin{aligned}
e^{2\mu}\tau^v_{\tilde{g}}(\sigma)_{a,i} &= \tau^v_g(\sigma)_{a,i} + (p-q)g(\nabla\mu, e_a)g(\nabla\mu, e_i) \\
&- 2g(\nabla\mu, e_a)g(H_\sigma, e_i) + 2g(\nabla\mu, e_i)g(H_{\sigma^\perp}, e_a) \\
&- 2g(\nabla_{e_i}(\nabla\mu)^\perp, e_a) + 2g(\nabla_{e_a}(\nabla\mu)^\top, e_i) \\
&+ (n-2)g(\nabla_{\nabla\mu}e_a, e_i).
\end{aligned}
\tag{3.5}
$$

where e_α is an orthonormal local frame such that $e_a \in \sigma$, $e_i \in \sigma^\perp$.

Proof. First we prove (3.4). Since $\tau^h(\sigma)$ is tensorial, we may work with the basis e_α. Let $\mu_a = e_a\mu$. By (3.2) we have

$$(\tilde{\nabla}_{e_b}e_a)^\perp = (\nabla_{e_b}e_a)^\perp - \delta_{ab}(\nabla\mu)^\perp.$$

Thus

$$
\begin{aligned}
\sum_{a,b}\tilde{R}((\tilde{\nabla}_{e_b}e_a)^\perp, e_a)e_a &= \sum_{a,b}\tilde{R}((\nabla_{e_b}e_a)^\perp, e_a)e_a - \sum_a \tilde{R}((\nabla\mu)^\perp, e_a)e_a \\
&= S_\sigma - T_\sigma.
\end{aligned}
$$

Since $\sum_a \mu_a^2 = |(\nabla\mu)^\top|^2$ and $\sum_a \mathrm{hess}_\mu((\nabla\mu)^\perp, e_a)e_a = (\mathrm{Hess}_\mu((\nabla\mu)^\perp))^\top$, we have by (3.3)

$$
\begin{aligned}
T_\sigma &= \sum_a R((\nabla\mu)^\perp, e_a)e_a - \sum_a \mathrm{Hess}_\mu((\nabla\mu)^\perp) + (\sum_a |(\nabla\mu)^\perp|^2)\nabla\mu \\
&+ \sum_a(\mu_a^2 - |\nabla\mu|^2 - \mathrm{hess}_\mu(e_a, e_a))(\nabla\mu)^\perp \\
&- \sum_a(|(\nabla\mu)^\perp|^2\mu_a - \mathrm{hess}_\mu((\nabla\mu)^\perp, e_a)e_a \\
&= \mathrm{Ric}_\sigma((\nabla\mu)^\perp) - p\mathrm{Hess}_\mu((\nabla\mu)^\perp) + p|(\nabla\mu)^\perp|^2\nabla\mu \\
&+ (|(\nabla\mu)^\top|^2 - p|\nabla\mu|^2 - \Delta_\sigma\mu)(\nabla\mu)^\perp \\
&- |(\nabla\mu)^\perp|^2(\nabla\mu)^\top + (\mathrm{Hess}_\mu((\nabla\mu)^\perp))^\top.
\end{aligned}
$$

Moreover, since second fundamental form is tensorial

$$
\begin{aligned}
\sum_{a,b}\mu_a\mu_b(\nabla_{e_b}e_a)^\perp &= (\nabla_{(\nabla\mu)^\top}(\nabla\mu)^\top)^\perp, \\
\sum_{a,b}\mathrm{hess}_\mu(e_a, e_b)(\nabla_{e_b}e_a)^\perp &= \sum_b(\nabla_{e_b}(\nabla_{e_b}\nabla\mu)^\top)^\perp, \\
\sum_{a,b}g(\nabla\mu, (\nabla_{e_b}e_a)^\perp)\mu_b e_a &= -(\nabla_{(\nabla\mu)^\top}(\nabla\mu)^\perp)^\top, \\
\sum_{a,b}\mathrm{hess}_\mu((\nabla_{e_b}e_a)^\perp, e_b)e_a &= -\sum_b(\nabla_{e_b}(\nabla_{e_b}\nabla\mu)^\perp)^\top.
\end{aligned}
$$

Hence, again by (3.3)

$$
\begin{aligned}
S_\sigma &= \sum_{a,b} R((\nabla_{e_b} e_a)^\perp, e_a)e_b - \sum_a \mathrm{Hess}_\mu((\nabla_{e_a} e_a)^\perp)\\
&+ \sum_{a,b} \mu_a\mu_b(\nabla_{e_b} e_a)^\perp - \sum_a |\nabla\mu|^2(\nabla_{e_a} e_a)^\perp - \sum_{a,b} \mathrm{hess}_\mu(e_a, e_b)(\nabla_{e_b} e_a)^\perp\\
&- \sum_{a,b} g(\nabla\mu, (\nabla_{e_b} e_a)^\perp)\mu_b e_a + \sum_{a,b} \mathrm{hess}_\mu((\nabla_{e_b} e_a)^\perp, e_b)e_a\\
&+ \sum_a g((\nabla_{e_a} e_a)^\perp, \nabla\mu)\nabla\mu\\
&= \sum_{a,b} R((\nabla_{e_b} e_a)^\perp, e_a)e_b - \mathrm{Hess}_\mu(H^\sigma) + (\nabla_{(\nabla\mu)^\top}(\nabla\mu)^\top)^\perp\\
&- |\nabla\mu|^2 H^\sigma - \sum_b (\nabla_{e_b}(\nabla_{e_b}\nabla\mu)^\top)^\perp + (\nabla_{(\nabla\mu)^\top}(\nabla\mu)^\perp)^\top\\
&- \sum_b (\nabla_{e_b}(\nabla_{e_b}\nabla\mu)^\perp)^\top + g(H^\sigma, \nabla\mu)\nabla\mu.
\end{aligned}
$$

Analogously, by symmetry, we get that $T_{\sigma^\perp}$ and $S_{\sigma^\perp}$ for $\sigma^\perp$ are equal to

$$
\begin{aligned}
T_{\sigma^\perp} &= \mathrm{Ric}_{\sigma^\perp}((\nabla\mu)^\top) - q\mathrm{Hess}_\mu((\nabla\mu)^\top) + q|(\nabla\mu)^\top|^2\nabla\mu\\
&+ (|(\nabla\mu)^\perp|^2 - q|\nabla\mu|^2 - \Delta_{\sigma^\perp}\mu)(\nabla\mu)^\top\\
&- |(\nabla\mu)^\top|^2(\nabla\mu)^\perp + (\mathrm{Hess}_\mu((\nabla\mu)^\top))^\perp.
\end{aligned}
$$

and

$$
\begin{aligned}
S_{\sigma^\perp} &= \sum_{i,j} R((\nabla_{e_j} e_i)^\top, e_i)e_j - \mathrm{Hess}_\mu(H^{\sigma^\perp}) + (\nabla_{(\nabla\mu)^\perp}(\nabla\mu)^\perp)^\top\\
&- |\nabla\mu|^2 H^{\sigma^\perp} - \sum_j (\nabla_{e_j}(\nabla_{e_j}\nabla\mu)^\perp)^\top + (\nabla_{(\nabla\mu)^\perp}(\nabla\mu)^\top)^\perp\\
&- \sum_j (\nabla_{e_j}(\nabla_{e_j}\nabla\mu)^\top)^\perp + g(H^{\sigma^\perp}, \nabla\mu)\nabla\mu.
\end{aligned}
$$

Finally, $e^{4\mu}\tau^h_{\tilde{g}}(\sigma) = S_\sigma - T_\sigma + S_{\sigma^\perp} - T_{\sigma^\perp}$, so by above calculations (3.4) holds.

Now we prove (3.5). Take an orthonormal basis $f_\alpha = e^{-\mu}e_\alpha$ for $\tilde{g}$. Then

$$\tilde{g}(\tilde{\nabla}_{f_b} f_c, f_a) = e^{-\mu}(g(\nabla_{e_b} e_c, e_a) + \mu_c\delta_{ab} - \mu_a\delta_{bc}), \tag{3.6}$$

$$\tilde{g}(\tilde{\nabla}_{f_b} f_c, f_i) = e^{-\mu}(g(\nabla_{e_b} e_c, e_i) - \mu_i\delta_{bc}), \tag{3.7}$$

Put

$$
\begin{aligned}
P_1 &= 2e^{2\mu}\sum_{b,c} \tilde{g}(\tilde{\nabla}_{f_b} f_c, f_a)\tilde{g}(\tilde{\nabla}_{f_b} f_c, f_i),\\
P_2 &= 2e^{2\mu}\sum_{j,k} \tilde{g}(\tilde{\nabla}_{f_j} f_k, f_i)\tilde{g}(\tilde{\nabla}_{f_j} f_k, f_a),
\end{aligned}
$$

and

$$Q_1 = e^{2\mu} \sum_b \tilde{g}(\tilde{\nabla}_{f_b} \tilde{\nabla}_{f_b} f_a, f_i),$$
$$Q_2 = e^{2\mu} \sum_j \tilde{g}(\tilde{\nabla}_{f_j} \tilde{\nabla}_{f_j} f_i, f_a),$$

and

$$S_1 = e^{2\mu} \sum_b \tilde{g}(\tilde{\nabla}_{\tilde{\nabla}_{f_b} f_b} f_a, f_i),$$
$$S_2 = e^{2\mu} \sum_j \tilde{g}(\tilde{\nabla}_{\tilde{\nabla}_{f_j} f_j} f_i, f_a),$$

Then, by (3.2) using (3.6) and (3.7) we get

$$P_1 = \sum_{b,c} g(e_a, \nabla_{e_b} e_c) g(\nabla_{e_b} e_c, e_i) - \mu_i g(\sum_b \nabla_{e_b} e_b, e_a) + g(\nabla_{e_a} (\nabla\mu)^\top, e_i) - \mu_a g(H^\sigma, e_i) + (p-1)\mu_a \mu_i$$

and

$$Q_1 = \sum_b g(\nabla_{e_b} \nabla_{e_b} e_a, e_i) + (2-p)\mu_a \mu_i - g(\nabla_{(\nabla\mu)^\top} e_a, e_i) + \mu_i g(\sum_b \nabla_{e_b} e_b, e_a) + \mu_a g(H^\sigma, e_i) - \mathrm{hess}_\mu(e_a, e_i)$$

and

$$S_1 = \sum_b g(\nabla_{\nabla_{e_b} e_b} e_a, e_i) + \mu_a g(H^\sigma, e_i) - \mu_a \mu_i - \mu_i g(\sum_b \nabla_{e_b} e_b, e_a) + g(\nabla_{(\nabla\mu)^\top} e_a, e_i) - p g(\nabla_{\nabla\mu} e_a, e_i).$$

Analogously, interchanging i with a, b with j and $\top$ with $\perp$, we get

$$P_2 = \sum_{j,k} g(e_a, \nabla_{e_j} e_k) g(\nabla_{e_j} e_k, e_i) - \mu_a g(\sum_j \nabla_{e_j} e_j, e_i) + g(\nabla_{e_i} (\nabla\mu)^\perp, e_a) - \mu_i g(H^{\sigma^\perp}, e_a) + (q-1)\mu_a \mu_i$$

and

$$Q_2 = \sum_j g(\nabla_{e_j} \nabla_{e_j} e_i, e_a) + (2-q)\mu_a \mu_i - g(\nabla_{(\nabla\mu)^\perp} e_i, e_a) + \mu_a g(\sum_j \nabla_{e_j} e_j, e_i) + \mu_i g(H^{\sigma^\perp}, e_a) - \mathrm{hess}_\mu(e_i, e_a)$$

and

$$S_2 = \sum_j g(\nabla_{\nabla_{e_j} e_j} e_i, e_a) + \mu_i g(H^{\sigma^\perp}, e_a) - \mu_a \mu_i - \mu_a g(\sum_j \nabla_{e_j} e_j, e_i) + g(\nabla_{(\nabla\mu)^\perp} e_i, e_a) - q g(\nabla_{\nabla\mu} e_i, e_a).$$

Since $e^{2\mu}\tau^v_{\tilde g}(\sigma)_{a,i} = P_1 - P_2 + (Q_1 - S_1) + (Q_2 - S_2)$, (3.5) holds. □

Corollary 3.4. *If σ and $\sigma^\perp$ are 1–dimensional foliations, then vertical harmonicity depends only on the conformal structure of M.*

Proof. Let $\sigma = \mathrm{Span}(X)$, $\sigma^\perp = \mathrm{Span}(Y)$, where X, Y is an orthonormal frame on M. Let $\mu_X = g(\nabla\mu, X)$, $\mu_Y = g(\nabla\mu, Y)$ and let $H^\sigma = h_\sigma Y$, $H^{\sigma^\perp} = h_{\sigma^\perp} X$. Then condition (3.5) is

$$\begin{aligned}
e^{2\mu}\tau^v_{\tilde g}(\sigma) &= \tau^v_g(\sigma) - 2\mu_X h_\sigma + 2\mu_Y h_{\sigma^\perp} \\
&\quad - 2g(\nabla_Y(\mu_Y Y), X) + 2g(\nabla_X(\mu_X X), Y) \\
&= \tau^v_g(\sigma) - 2\mu_X h_\sigma + 2\mu_Y h_{\sigma^\perp} - 2\mu_Y g(\nabla_Y Y, X) + 2\mu_X g(\nabla_X X, Y) \\
&= \tau^v_g(\sigma).
\end{aligned}$$

Thus condition $\tau^v_g(\sigma) = 0$ depends only on the conformal structure induced by g. □

The following result states that for totally geodesic foliations of the same dimension vertical harmonicity is also a conformal invariant.

Corollary 3.5. *If σ and $\sigma^\perp$ are totally geodesic foliations with respect to g and $\dim\sigma = \dim\sigma^\perp$, then σ and $\sigma^\perp$ are vertically harmonic with respect to $\tilde g = e^{2\mu}$ for any μ.*

Proof. Since σ and $\sigma^\perp$ are totally geodesic, we have

$$g(\nabla_{e_i}(\nabla\mu)^\perp, e_a) = g(\nabla_{e_a}(\nabla\mu)^\top, e_i) = 0, \quad H^\sigma = H^{\sigma^\perp} = 0$$

and

$$g(\nabla_{\nabla\mu} e_a, e_i) = g(\nabla_{(\nabla\mu)^\top} e_a, e_i) - g(\nabla_{(\nabla\mu)^\perp} e_i, e_a) = 0$$

and by Proposition 3.2 $\tau^v_g(\sigma)_{a,i} = 0$. Hence by (3.5) $\tau^v_{\tilde g}(\sigma)_{a,i} = 0$. □

Now we describe the horizontal tension field for manifolds of constant curvature. Define an operator $W : TM \to TM$ by

$$W(X) = X^\top - X^\perp.$$

Then

Corollary 3.6. *Assume M is of nonzero constant curvature κ, $\dim\sigma = \dim\sigma^\perp = n/2$. Then for $\tilde g = e^{-2\mu} g$, we have*

$$\begin{aligned}
e^{4\mu}\tau^h_{\tilde g}(\sigma) &= (\kappa - |\nabla\mu|^2)H + (g(\nabla\mu, H) - \frac{n}{2}\kappa)\nabla\mu \\
&\quad + (\Delta_\sigma\mu)(\nabla\mu)^\perp + (\Delta_{\sigma^\perp}\mu)(\nabla\mu)^\top \\
&\quad - \mathrm{Hess}_\mu(H - \frac{n}{2}\nabla\mu) + W([(\nabla\mu)^\top, (\nabla\mu)^\perp]) \\
&\quad - \mathrm{tr}(\nabla_*(\nabla_*\nabla\mu)^\perp)^\top - \mathrm{tr}(\nabla_*(\nabla_*\nabla\mu)^\top)^\perp.
\end{aligned} \tag{3.8}$$

Proof. We have

$$\mathrm{Ric}_\sigma((\nabla\mu)^\perp) = \kappa p(\nabla\mu)^\perp, \quad \mathrm{Ric}_{\sigma^\perp}((\nabla\mu)^\top) = \kappa q(\nabla\mu)^\top$$

and by Proposition 3.2, $\tau^h_g(\sigma) = \kappa H$. Hence (3.4) takes the form (3.8). □

Corollary 3.4 is analogous to the general fact that harmonicity of a map from 2–dimensional manifold depends only on the conformal structure (see [1, Corollary 3.5.4]). In the example below we show that similar condition for horizontal harmonicity does not hold.

Example 3.7. Let σ and $\sigma^\perp$ be foliations by lines in Euclidean space $\mathbb{R}^2$. With coordinates (x, y), $\sigma = \{y = \mathrm{const}\}$, $\sigma^\perp = \{x = \mathrm{const}\}$. Let $\mu : \mathbb{R}^2 \to \mathbb{R}$ and denote the derivatives $\partial\mu/\partial x$ and $\partial\mu/\partial y$ by μ'_x and μ'_y respectively. By Corollary 3.5 σ and $\sigma^\perp$ are vertically harmonic with respect to $\tilde{g} = e^{2\mu}g$. Since the curvature tensor $R = 0$, then σ and $\sigma^\perp$ are horizontally harmonic with respect to g. Let $\tau^h_{\tilde{g}}(\sigma) = (\tau^h_{\tilde{g}}(\sigma)_x, \tau^h_{\tilde{g}}(\sigma)_y)$. Then condition (3.4) simplifies to

$$\begin{aligned} \tau^h_{\tilde{g}}(\sigma)_x &= \mu'_x \Delta\mu, \\ \tau^h_{\tilde{g}}(\sigma)_y &= \mu'_y \Delta\mu. \end{aligned}$$

Hence, σ and $\sigma^\perp$ are harmonic with respect to $\tilde{g}$ if and only if $\Delta\mu = 0$ i.e. μ is a harmonic function.

The fact that horizontal harmonicity is not a conformal invariant in the case $\dim M = 2$, $\dim\sigma = \dim\sigma^\perp = 1$ can be deduced differently. Foliations in the Example 3.7 are harmonic and minimal with respect to Euclidean metric $\langle\cdot,\cdot\rangle$, but with the conformal metric $\tilde{g} = 4/(1+r^2)^2\langle\cdot,\cdot\rangle$, $r^2 = x^2+y^2$, of constant curvature $\kappa = 1$ they are not minimal. Hence by Proposition 3.1 they are not harmonic. Compare also the following example.

Example 3.8. We will give an example to Corollary 3.6. Consider a plane without origin $M = \mathbb{R}^2 \setminus \{0\}$ and define a Riemannian metric g on M by

$$g(x, y) = \frac{4}{(1 + x^2 + y^2)^2}\langle\cdot,\cdot\rangle, \quad (x, y) \in M,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product. Then M is of constant sectional curvatue $\kappa = 1$. Let $r^2 = x^2 + y^2$ and define two vector fields

$$\begin{aligned} X &= \frac{1 + r^2}{2r}(-y\partial_x + x\partial_y), \\ Y &= \frac{1 + r^2}{2r}(x\partial_x + y\partial_y). \end{aligned}$$

Then X, Y form an orthonormal basis with respect to g. Let $\sigma = \mathrm{Span}(X)$, $\sigma^\perp = \mathrm{Span}(Y)$. Obviously, σ is the foliation by circles, $\sigma^\perp$ the foliation by rays. Moreover,

$$\nabla_X X = \frac{r^2-1}{2r}Y, \quad \nabla_X Y = \frac{1-r^2}{2r}X, \quad \nabla_Y X = \nabla_Y Y = 0.$$

Therefore, $H^\sigma = \frac{r^2-1}{2r}Y$, $H^{\sigma^\perp} = 0$.

Let $\tilde{g} = e^{2\mu}g$ for some smooth function μ. We will seek for a solution when $\mu = \mu(r)$ is a function of a radius r. Then $X\mu = 0$. Therefore condition (3.8) simplifies to the following differential equation

$$0 = (1-(Y\mu)^2)A - Y\mu + (Y\mu)A^2 - (Y^2\mu)A + (Y\mu)(Y^2\mu), \quad A = \frac{r^2-1}{2r}.$$

Putting $f = Y\mu$, by the fact that $Y = \frac{r^2+1}{2}\frac{\partial}{\partial r}$, we get first order ordinary differential equation

$$0 = 2r(r^2-1)(1-f^2) + ((r^2-1)^2-4r^2)f - r(r^4-1)f' + 2r^2(r^2+1)ff'. \tag{3.9}$$

Solving (3.9) as a quadratic equation with respect to f we get

$$f(r) = \frac{r^2-1}{2r} \quad \text{or} \quad r(r^2+1)f' = (r^2-1)f + 2r.$$

The solutions to above equation are

$$f = \frac{C(r^2+1)-1}{r}.$$

Hence μ is of the form

$$\mu = \log(D(r^2+1)r^{2(C-1)}), \quad e^{2\mu} = D^2(r^2+1)^2 r^{4(C-1)},$$

where C, D are constants, $D > 0$. For $C = 1$, $D = 1/2$ we get Euclidean metric and it agrees with the fact that in the Euclidean space any distribution is horizontally harmonic, since then curvature tensor vanishes.

Easy computations show that $\tau^v_g(\sigma) = 0$, hence by Proposition 2.1 and Corollary 3.4, σ and $\sigma^\perp$ are vertically harmonic with respect to any Riemannian metric conformal to g. Finally, σ and $\sigma^\perp$ are harmonic with respect to Riemannian metrics

$$\tilde{g} = D^2(r^2+1)^2 r^{4(C-1)} g = 4D^2 r^{4(C-1)}\langle \cdot, \cdot \rangle, \quad C, D \in \mathbb{R}, D > 0.$$

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